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COMMENT

Soliton excitation in coupled complex scalar field theory

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Abstract. Sarker *et al* have obtained two soliton solutions for a complex one-dimensional field. We present a simple and effective method for obtaining the exact soliton solutions of the same equation. The method suggested here can be applied to other coupled non-linear systems as well.

Most of the relatively few known analytic solitary wave solutions are to non-linear wave equations for a real scalar field in one space and one time dimension. Sarker *et al* (1976) considered that a certain equation for a complex scalar field in one dimension arises from the particular context of weakly pinned charge-density-wave condensates and presented soliton solutions in the particular case. In this paper, we find the analytic soliton solutions of the same equation by a simple and effective method. One considers the Lagrangian density as follows (Sarker *et al* 1976):

$$\mathcal{L} = \tfrac{1}{2}|\psi_t|^2 - \tfrac{1}{2}c^2|\psi_x|^2 + \tfrac{1}{2}a|\psi|^2 - \tfrac{1}{4}b|\psi|^4 - d|\psi|^2[1 - \cos(2\varphi)] \qquad (1)$$

where $\psi(x,t) = u(x,t)\exp[\mathrm{i}\varphi(x,t)]$ is a complex field. The last three terms in equation (1) can be thought of as the local potential $V(\psi) = -\frac{1}{2}au^2 + \frac{1}{4}bu^4 + du^2[1-\cos(2\varphi)]$ whose continuous rotational symmetry has been broken and replaced with a (twofold) discrete symmetry. The amount of coupling in both u and φ should clearly depend on the relative magnitudes of the central hump and phase 'dimple' potentials, or the ratio d/a (Bishop and Schneider 1978).

We rewrite equation (1) for the real and imaginary parts of the wavefunction $\psi = \sigma + \mathrm{i}\rho$ and the Lagrangian density becomes

$$\mathcal{L} = \tfrac{1}{2}(\sigma_t^2 + \rho_t^2) - \tfrac{1}{2}c^2(\sigma_x^2 + \rho_x^2) + \tfrac{1}{2}a(\sigma^2 + \rho^2) - \tfrac{1}{4}b(\sigma^2+\rho^2)^2 - 2d\rho^2. \qquad (2)$$

The coupled non-linear equations of motion are given by

$$\sigma_{tt} - c^2\sigma_{xx} - a\sigma + b\sigma(\sigma^2+\rho^2) = 0 \qquad (3a)$$

$$\rho_{tt} - c^2\rho_{xx} - (a-4d)\rho + b\rho(\sigma^2+\rho^2) = 0. \qquad (3b)$$

One seeks soliton solutions to equation (3) of the form $\sigma = \sigma(s)$, $\rho = \rho(s)$, where $s = (r/c)(x - vt)$ with $r = (1 - v^2/c^2)^{-1/2}$. The equations of motion are

$$\sigma_{ss} = -a\sigma + b\sigma(\sigma^2+\rho^2) \qquad (4a)$$

$$\rho_{ss} = (4d - a)\rho + b\rho(\sigma^2+\rho^2). \qquad (4b)$$

Note that, because of the symmetry of equation (4), one carries out the non-linear transformation

$$\sigma_s = A\sigma\rho \tag{5a}$$

$$\rho_s = B + C\rho^2 + D\sigma^2. \tag{5b}$$

From equation (5), we have

$$\sigma_{ss} = AB\sigma + AD\sigma^3 + (A^2 + AC)\sigma\rho^2 \tag{6a}$$

$$\rho_{ss} = 2CB\rho + 2C^2\rho^3 + 2(CD + AD)\rho\sigma^2. \tag{6b}$$

Equations (4) and (6) possess the same mathematical structure, which implies that equation (6) is completely equal to equation (4) if equation (5) is a correct non-linear transformation. On comparison of equation (4) with equation (6), the real constants A, B, C and D satisfy the following relations:

$$\begin{aligned} &AB = -a \qquad AD = A^2 + AC = b \\ &2CB = (4d - a) \qquad 2C^2 = 2(CD + AD) = b. \end{aligned} \tag{7}$$

Equations (7) give

$$A = a\sqrt{2b}/(a - 4d) \quad B = (4d - a)/\sqrt{2b} \quad C = \sqrt{b/2}$$
$$D = b(a - 4d)/a\sqrt{2b} \quad a = 2d \tag{8a}$$

and

$$A = a\sqrt{2b}/(4d - a) \quad B = (a - 4d)/\sqrt{2b} \quad C = -\sqrt{b/2}$$
$$D = b(4d - a)/a\sqrt{2b} \quad a = 2d. \tag{8b}$$

It needs to be pointed out that the symmetry of equation (4) means that the non-linear transformations (5) can only be used if the coefficients of the original Lagrangian satisfy the condition $a = 2d$. Substituting (5*a*) into (5*b*) and using (8), we immediately obtain

$$\sigma_{ss} - \sigma_s^2/2\sigma = -a\sigma + b\sigma^3. \tag{9}$$

Eliminating the non-linear term $-\sigma_s^2/2\sigma$ from equation (9) with the aid of the transformation $\sigma = y^m$, equation (9) becomes

$$y_{ss} = -ay/m + by^{2m+1}/m \tag{10}$$

with $m = 2$.

It is easy to obtain the solution of equation (10) as follows:

$$\sigma = \pm\sqrt{-3a/b}\,\mathrm{cosech}[\sqrt{-2a}(r/c)(x - vt) + C_0] \tag{11a}$$

$$\rho = \pm\sqrt{-a/b}\,\mathrm{cotanh}[\sqrt{-2a}(r/c)(x - vt) + C_0] \tag{11b}$$

where $r = (1 - v^2/c^2)^{-1/2}$ and C_0 is an arbitrary real constant.

As equations (4) possess a certain symmetrical structure, thus we may perform another non-linear transformation

$$\sigma_s = \bar{A} + \bar{B}\sigma^2 + \bar{C}\rho^2 \tag{12a}$$

$$\rho_s = \bar{D}\sigma\rho. \tag{12b}$$

Similarly we obtain another soliton solution of equation (4) by using the same method as follows:

$$\sigma = \pm\sqrt{a/b}\,\mathrm{cotanh}[\sqrt{2a}(r/c)(x - vt) + C_0] \tag{13a}$$

$$\rho = \pm\sqrt{3a/b}\,\mathrm{cosech}[\sqrt{2a}(r/c)(x - vt) + C_0] \tag{13b}$$

where r and C_0 are the same as above.

For us it is convenient to go over to the Hamiltonian formalism. The corresponding Hamiltonian density is

$$\mathcal{H} = \tfrac{1}{2}(\sigma_t^2 + \rho_t^2) + \tfrac{1}{2}c^2(\sigma_x^2 + \rho_x^2) - \tfrac{1}{2}a(\sigma^2 + \rho^2) + \tfrac{1}{4}b(\sigma^2 + \rho^2)^2 + 2d\rho^2. \tag{14}$$

The corresponding soliton energy is given by (here we adopt the renormalization technique)

$$E = \int_{-\infty}^{+\infty} [\mathcal{H} - \mathcal{H}(\sigma(\infty), \rho(\infty))]\,\mathrm{d}x. \tag{15}$$

One can derive expressions for the energy of both solitons from equations (15), (11) and (13); it is straightforward to show that

$$E_1 = -26a^2[r^2(v^2 + c^2) + c^2]/3\sqrt{-2a}bcr \tag{16a}$$

$$E_2 = -26a^2[r^2(v^2 + c^2) + c^2]/3\sqrt{2a}bcr. \tag{16b}$$

From equation (16), we also have the effective masses of both solitons:

$$m_1^* = -52a^2/3\sqrt{-2a}bc \qquad (a < 0, b > 0, c < 0) \tag{17a}$$

$$m_2^* = -52a^2/3\sqrt{2a}bc \qquad (a > 0, b > 0, c < 0). \tag{17b}$$

In conclusion, we have found two kinds of solitary-wave solutions to certain coupled equations of motion for a complex scalar field in one dimension. The stable solutions of equations (3) and the exact soliton solutions of the coupled relativistic scalar field theory (Rajaraman 1979) may be included in the same category. Generally a relevant transformation is powerful for solving the related non-linear problems. It is well known that the Cope–Hopf transformation (5*a*), etc, are useful transformations which play an important role in the linearization of non-linear problems such as the Burgers equation. The method suggested here can be applied to non-linear diffusion systems such as the Fisher equation (Wang 1988) and to coupled non-linear differential equations systems; for example, we may obtain soliton solutions for the general coupled non-linear equations with damped terms by the above method:

$$\begin{aligned} r_1\,\partial u/\partial t + \partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 &= f_1(u, v) \\ r_2\,\partial v/\partial t + \partial^2 v/\partial t^2 - \partial^2 v/\partial x^2 &= f_2(u, v) \end{aligned} \tag{18}$$

where r_1 and r_2 are damped coefficients and $f_1(u, v)$ and $f_2(u, v)$ are analytic function for u and v. Let $u(x,t)=u(x-ct)=u(s)$, $v(x,t)=v(x-ct)=v(s)$; then equation (18) becomes

$$\begin{aligned}-cr_1\,\mathrm{d}u/\mathrm{d}s+c^2\,\mathrm{d}^2u/\mathrm{d}s^2-\mathrm{d}^2u/\mathrm{d}s^2&=f_1(u,v)\\-cr_2\,\mathrm{d}v/\mathrm{d}s+c^2\,\mathrm{d}^2v/\mathrm{d}s^2-\mathrm{d}^2v/\mathrm{d}s^2&=f_2(u,v).\end{aligned}\tag{19}$$

If we take

$$\mathrm{d}u/\mathrm{d}s=P_1(u,v)\qquad \mathrm{d}v/\mathrm{d}s=P_2(u,v)\tag{20}$$

equation (19) can be written as

$$\begin{aligned}-cr_1P_1+(c^2-1)[(\partial P_1/\partial u)P_1+(\partial P_1/\partial v)P_2]&=f_1(u,v)\\-cr_2P_2+(c^2-1)[(\partial P_2/\partial u)P_1+(\partial P_2/\partial v)P_2]&=f_2(u,v).\end{aligned}\tag{21}$$

Looking for suitable non-linear functions $P_1(u,v)$ and $P_2(u,v)$ which satisfy equation (21) and then integrating equation (20), we may derive solutions of equation (18).

Furthermore, we may perform a linear stability analysis of the soliton solutions, as the stability of the soliton solution needs to be considered in the practical physical process.

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